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INSTITUTE OF MATHEMATICAL SCIENCES
25 Waverly Place, New York 3, N. Y.

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BEN NOBLE

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THE POTENTIAL AND CHARGE DISTRIBUTION NEAR THE TIP OF A FLAT ANGULAR SECTOR

Ben Noble

Ben Noble.

Ben Noble

Sidney Borowitz

Sidney Borowitz
Acting Director

Dr. Werner W. Gerbes

Dr. Werner Gerbes
Contract Monitor

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Abstract

The calculation of the potential and charge distribution near the tip of a flat angular sector is reduced to the solution of a mixed boundary value problem on the surface of a sphere. A method is developed for finding the lowest eigenvalue and eigenfunction for this mixed problem. If r is the distance from the tip of the sector and the potential varies as r^{μ} when r tends to zero, numerical results are given for μ as a function of the angle of the sector.

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I. Introduction

We consider solutions of the reduced scalar wave equation which vanish or have vanishing normal derivatives on a plane angular sector. The potential Ψ satisfies

$$(1) \qquad \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Psi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Psi}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 \Psi}{\partial \phi^2} + k^2 r^2 \Psi = 0 ,$$

where (r, θ, ϕ) are spherical polar coordinates. The sector will be assumed to lie in $0 < r < \infty$, $\theta = \pi/2$, $-\alpha < \phi < \alpha$. This orientation has been chosen for the convenience of the analysis given later.

The importance of this problem in electrostatics and diffraction theory is that such a sector is the simplest flat strip with a sharp corner.

In [5] it is reported via an oral communication that H. Lewy has established the following result. If the potential near the tip of a flat sector of angle 2α on which $\Psi = 0$ varies as r^μ for $r \to 0$ with fixed ϕ, θ, then $\partial\mu/\partial\alpha = 1/\pi$ for $\alpha = \pi/2$. ($\alpha = \pi/2$ corresponds to the case of a half-plane for which $\mu = 1/2$ exactly). It is also known that $\mu = 0$ for $\alpha = 0$ and $\mu = 1$ for $\alpha = \pi$. Prof. Lewy has conjectured that the relation $\mu = \alpha/\pi$ holds for $0 \leq \alpha \leq \pi$. Our numerical results are consistent with the linear variation for α near $\pi/2$ but disprove the conjecture that μ varies linearly with α over the whole range $0 \leq \alpha \leq \pi$.

A crude method for solution of the problem is discussed in [1]. Two expressions are suggested for the variation of μ with α which give $\partial\mu/\partial\alpha = 0.25$ and 0.50 at $\alpha = \pi/2$ compared with the exact result of $1/\pi = 0.32$. It is clear by comparison with our results in Section 5 below that the expressions obtained in [1] have approximate validity only for α near $\pi/2$.

A solution by separation-of-variables in a sphero-conal coordinate system has been obtained by Kraus [5]. This requires a rather involved theoretical analysis using Lamé functions followed by a determination of simultaneous eigenvalues if numerical results are required. Although the method used by Kraus is undoubtedly correct, there would seem to be errors in his numerical results. (See Section 5 and Table II below.)

The present analysis is essentially elementary.

II. Formulation

We use separation-of-variables in (1) by assuming

$$\Psi = R(r)\,\Phi(\theta,\phi) \ .$$

Then R(r) satisfies

$$\frac{d}{dr}\, r^2\, \frac{dR}{dr} + \left[k^2 r^2 - \mu(\mu+1)\right] R = 0 \ ,$$

where the separation parameter has been written as $\mu(\mu+1)$ for convenience.

The solution of this equation is

$$R(r) = r^{-1/2} \left\{ A J_{\mu+1/2}(kr) + B Y_{\mu+1/2}(kr) \right\} \tag{2}$$

where a solution finite at the origin is given by B = 0, and a solution satisfying the radiation condition is given by B = iA. The function Φ then satisfies

$$\frac{1}{\sin\theta}\, \frac{\partial}{\partial\theta}\left(\sin\theta\, \frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\, \frac{\partial^2\Phi}{\partial\phi^2} + \mu(\mu+1)\,\Phi = 0. \tag{3}$$

We consider only eigenfunctions which are symmetrical about $\phi = 0, \pi$, so that

$$\partial\Phi/\partial\phi = 0 \text{ on } \phi = 0,\pi, \qquad (0 \le \theta \le 1/2\ \pi) \ . \tag{4}$$

If $\Psi = 0$ on the sector we consider only eigenfunctions which are symmetrical about $\theta = \pi/2$ and then the following mixed boundary conditions hold on $\theta = \pi/2$:

$$\Phi = 0 \ , \qquad (0 \le \phi < \alpha), \tag{5a}$$

(5b) $\partial\underline{\Psi}/\partial\theta = 0 , \qquad (\alpha < \phi \leq \pi) .$

If $\partial\Psi/\partial\theta = 0$ on the sector we consider only eigenfunctions which are odd about $\theta = \pi/2$, and then the following mixed boundary conditions hold on $\theta = 1/2\ \pi$:

(6a) $\partial\underline{\Psi}/\partial\theta = 0, \qquad (0 \leq \phi < \alpha) ,$

(6b) $\Psi = 0, \qquad (\alpha < \phi \leq \pi) .$

Clearly the problems represented by (5) and (6) are completely equivalent. In particular the eigenvalues for a sector of half-angle α on which $\Psi = 0$ are the same as those for a sector of half-angle $(\pi - \alpha)$ on which $\partial\Psi/\partial\theta = 0$.

By separation-of-variables the solution of (3) which satisfies (4) can be written

(7) $$\underline{\Psi}(\theta,\phi) = \sum_{0}^{\infty} a_m P_\mu^m(\cos\theta)\cos m\phi .$$

We shall now consider the extreme cases $\alpha = \pi$ and $\alpha = 0$. When $\alpha = \pi$ we have $\Psi = 0$ in the whole plane $\theta = 1/2\ \pi$, $0 \leq \phi \leq \pi$. The lowest eigenfunction will be independent of ϕ so that $m = 0$ and we wish to find μ so that $P_\mu(0) = 0$. By symmetry a solution is required which is odd about $\theta = \pi/2$ and finite at $\theta = 0, \pi$. Hence $\mu = 1$ for $\alpha = \pi$. When $\alpha = 0$ the symmetry of (5) and (6) shows that this is equivalent to requiring a solution such that $\partial\Psi/\partial\theta = 0$ on the whole plane $\theta = \pi/2$, $0 \leq \phi \leq \pi$. Again we must have independence of ϕ so that $m = 0$, but this time the eigenfunction must be even about $\theta = \pi/2$, so that $\mu = 0$ for $\alpha = 0$.

Application of the boundary condition (5) to the solution (7) gives the following equations:

(8a) $$\sum_{0}^{\infty} a_m\ P_\mu^m(0)\ \cos m\phi = 0 , \qquad (0 \leq \phi < \alpha) ,$$

$$(8b) \qquad \sum_{0}^{\infty} a_m \left\{ \frac{d}{dz} P_\mu^m(z) \right\}_{z=0} \cos m\phi = 0, \qquad (\alpha < \phi \leq \pi) \ .$$

where ([3], p.145)

$$(9a) \qquad P_\mu^m(0) = 2^m \pi^{-1/2} \cos\left[\tfrac{1}{2}\pi(\mu+m)\right] \Gamma(\tfrac{1}{2} + \tfrac{1}{2}\mu + \tfrac{1}{2}m) \Big/ \Gamma(1 + \tfrac{1}{2}\mu - \tfrac{1}{2}m) \ ,$$

$$(9b) \qquad \left\{ dP_\mu^m(z)/dz \right\}_{z=0} = 2^{m+1} \pi^{-1/2} \sin\left[\tfrac{1}{2}\pi(\mu+m)\right] \Gamma(1 + \tfrac{1}{2}\mu + \tfrac{1}{2}m) \Big/ \Gamma(\tfrac{1}{2} + \tfrac{1}{2}\mu - \tfrac{1}{2}m) \ .$$

Equations (8) are dual equations possessing solutions only for special values of μ, which are the eigenvalues we wish to find.

Without loss of generality we can take $\mu > -1/2$, since $P_\mu^m(z) = P_{-\mu-1}^m(z)$. On writing

$$(10) \qquad a_m \left\{ dP_\mu^m(z)/dz \right\}_{z=0} = b_m$$

the equations (8) become

$$(11a) \qquad \sum_{0}^{\infty} R_m(\mu) b_m \cos m\phi = 0, \qquad (0 \leq \phi < \alpha) \ ,$$

$$(11b) \qquad \sum_{0}^{\infty} b_m \cos m\phi = 0, \qquad (\alpha < \phi \leq \pi),$$

where, from (9), after some reduction,

$$(12) \qquad R_m(\mu) = \frac{\Gamma(\frac{1}{2} + \frac{1}{2}m + \frac{1}{2}\mu)\ \Gamma(\frac{1}{2}m - \frac{1}{2}\mu)}{\Gamma(1 + \frac{1}{2}m + \frac{1}{2}\mu)\ \Gamma(\frac{1}{2} + \frac{1}{2}m - \frac{1}{2}\mu)} \ .$$

$$(13a) \qquad \text{If } \mu = 0 \ , \qquad R_o(\mu) = 2/m \ , \qquad m = 1,2,3,\ldots$$

$$(13b) \qquad \text{If } 0 < \mu \leq 1, \qquad R_o(\mu) \approx 2/m \ , \qquad m = 2,3,\ldots$$

The validity of this approximation is discussed in Section 5 below. We write (11) in the form

$$(14a) \qquad \frac{1}{2} R_o(\mu) b_o + \sum_1^\infty \frac{1}{m} b_m \cos m\phi + \sum_1^\infty \left\{ \frac{1}{2} R_m(\mu) - \frac{1}{m} \right\} \cos m\phi = 0 , \quad (0 \leq \phi < \alpha) ,$$

$$(14b) \qquad b_o + \sum_1^\infty b_m \cos m\phi = 0 , \qquad (\alpha < \phi \leq \pi) .$$

An approximate solution of this system is obtained in Section 3 by ignoring the second infinite series on the left of (14a). In Section 4 second and third order approximations are obtained by taking into account one and two terms of this infinite series.

III. A first approximation

As indicated at the end of Section 2 we solve the following dual equations exactly:

$$(15a) \qquad \frac{1}{2} R_o(\mu) b_o + \sum_o^\infty \frac{1}{m} b_m \cos m\phi = 0, \quad (0 \leq \phi < \alpha) ,$$

$$(15b) \qquad b_o + \sum_o^\infty b_m \cos m\phi = 0 , \quad (\alpha < \phi \leq \pi) .$$

The values of μ for which solutions of these equations exist can be found by a method developed in [7]. Suppose that (15b) equals an (unknown) function $f(\phi)$ for $0 \leq \alpha < \phi$. Then

$$(16) \qquad b_o = \frac{1}{\pi} \int_o^\alpha f(u)du \; : \qquad b_m = \frac{2}{\pi} \int_o^\alpha f(u) \cos mu\, du .$$

Substitute in (15a). Then

$$(17) \qquad \frac{1}{4} R_o(\mu) \int_o^\pi f(u)du + \sum_1^\infty \frac{1}{m} \int_o^\alpha f(u) \cos mu\, du \cos m\phi = 0, \quad (0 \leq \phi < \alpha) .$$

A transformation due to Schwinger (see [6], p.48) is

$$(18a)\qquad \sum_1^\infty \frac{1}{m}\cos mu\cos m\phi = -\frac{1}{2}\ell n\left\{2|\cos u - \cos\phi|\right\}$$

$$= -\frac{1}{2}\ell n\, s + \sum_1^\infty \frac{1}{m}\cos m\xi\cos mx\ ,$$

where

$$(18b)\qquad \cos u = t + s\cos\xi\ ,$$

and t,s are chosen so that $u = 0, \alpha$ corresponds to $\xi = 0, \pi$, respectively.

This gives

$$(18c)\qquad t = \cos^2\frac{1}{2}\alpha \quad : \quad s = \sin^2\frac{1}{2}\alpha\ .$$

A similar relation is assumed between ϕ and x. Equation (17) becomes

$$(19)\qquad \frac{1}{2}\left\{\frac{1}{2}R_o(\mu) - \ell ns\right\}\int_o^\pi f(u)\frac{du}{d\xi}\,d\xi + \sum_1^\infty \frac{1}{m}\int_o^\pi f(u)\frac{du}{d\xi}\cos m\xi d\xi\cos mx = 0\ ,$$

$$(0 < x < \pi)\ .$$

This integral equation can be solved exactly. Choose

$$(20)\qquad f(u)du/d\xi = \text{constant} = C, \text{ say},$$

i.e.

$$(21)\qquad f(\phi) = C\sin\phi\left\{s^2 - (t - \cos\phi)^2\right\}^{-1/2}, \quad (0 \le \phi < \alpha)\ .$$

Then (19) is satisfied provided that

$$(22)\qquad R_o(\mu) = 2\ell n\, s = 4\ell n(\sin\tfrac{1}{2}\alpha)\ .$$

This is a transcendental equation for the determination of μ. The solutions of this equation such that $0 < \mu < 1$, for various α, are given in Table II (see Section 5 below).

By definition $f(\phi) = -|\partial\underline{\Psi}/\partial\theta|$ at $\theta = \frac{1}{2}\pi$. The charge distribution on the face of the sector, for $\Psi = 0$ on the sector, is proportional to $r^{-1}\partial\Psi/\partial\theta = r^{-1}R(r)\,\partial\underline{\Psi}/\partial\theta$, on $\theta = \frac{1}{2}\pi$ i.e. near the tip, from (2), (21), to

$$\left[Ar^{\mu-1} + Br^{-\mu-2}\right] \frac{\sin \phi}{\left\{s^2 - (t-\cos \phi)\right\}^{1/2}} , \quad (0 < \phi < \alpha) .$$

If r is fixed, $\phi \to \alpha-0$, and R is the distance from (r,ϕ) to $(r,\frac{1}{2}\alpha)$, $\phi < \alpha$, then the charge distribution behaves as $R^{-1/2}$ as $R \to 0$, as it should do. If ϕ is fixed and $r \to 0$ then the charge distribution behaves as $(Ar^{\mu-1} + Br^{-\mu-2})$. This tells us incidentally that if $\alpha = \frac{1}{2}\pi$, then we should obtain $\mu = 1/2$, since in this case the sector reduces to a half-plane (cf. Table II and Section 5).

Another quantity of physical interest is the value of Ψ on $\theta = \frac{1}{2}\pi$ for $\alpha \leq \phi \leq \pi$. On retracing our steps we find that to the order of approximation used in this section

$$2\pi \underline{\Psi}(\tfrac{1}{2}\pi,\phi) = -R_o(\mu)\int_o^\alpha f(u)du - 4\sum_1^\infty \frac{1}{m}\int_o^\alpha f(u)\cos mu\,du\cos m\phi$$

$$= -R_o(\mu)\int_o^\alpha f(u)du + 2\int_o^\alpha f(u)\ell n(2|\cos u - \cos\phi|)du ,$$

where we have used (18a). Introduce the change of variable (18b) for u. We are now interested in $\alpha \leq \phi \leq \pi$ so that it is not possible to introduce the corresponding change of variable for ϕ. This gives

$$\pi \underline{\Psi}(\tfrac{1}{2}\pi,\phi) = C\int_o^\pi \ell n(2|\cos\xi + A|)d\xi ,$$

where we have used the solution (20) and the result (22). Also

$$A = s^{-1}(t - \cos\phi) = 1 + (\cos\alpha - \cos\phi)(\sin\tfrac{1}{2}\alpha)^{-2}.$$

Since $\phi > \alpha$ we have $A > 1$. If we introduce $\underline{a}$ defined by

(23) $$(1 + a^2)/(2a) = A, \qquad a < 1,$$

we have ([2], formula 418)

$$\ell n\, 2(A+\cos\xi) = -\ell n\, a + 2(a\cos\xi - \tfrac{1}{2}a^2\cos 2\xi + \tfrac{1}{3}a^3\cos 3\xi_\xi - \ldots)$$

Hence

$$\pi \underline{\Psi}(\tfrac{1}{2}\pi,\ \phi) = -\ell n\, a.$$

By solving the quadratic equation (23) and choosing the root which is less than unity we find

$$a = \left\{ \frac{\sin \frac{1}{2}\phi - \left[\sin \frac{1}{2}(\phi + \alpha)\sin \frac{1}{2}(\phi - \alpha)\right]^{1/2}}{\sin \frac{1}{2}\alpha} \right\}^2 .$$

The value of Ψ on $\theta = \frac{1}{2}\pi$, $\alpha < \phi < \pi$ is therefore proportional to

$$-(Ar^{\mu} + Br^{-\mu-1})\ell n\, a, \qquad \text{(for small r)}.$$

If r is fixed, $\phi \to \alpha + 0$, and R is the distance from (r,ϕ) to (r,α), $\phi > \alpha$, then Ψ behaves as $R^{1/2}$ as $R \to 0$, as it should do. If ϕ is fixed and $r \to 0$ then Ψ behaves like $(Ar^{\mu} + Br^{-\mu-1})$. As before we can deduce that if $\theta = \frac{1}{2}\pi$ then $\mu = \frac{1}{2}$.

An approximation to the eigenfunction corresponding to the lowest eigenvalue can be found in the following way. On using the value of $f(\phi)$ from (21) in (16) we can determine the constants b_m and hence the a_m from (10). The eigenfunction is then given by (7) using the μ determined from (22).

IV. Second and third approximations

Instead of using (15a) as an approximation for (11a) we can include the effect of $R_1(\mu)$ by solving

$$\frac{1}{2}R_o(\mu)b_o + \left\{\frac{1}{2}R_1(\mu) - 1\right\} b_1 \cos\phi + \sum_1^{\infty} \frac{1}{m} b_m \cos m\phi = 0 , \quad (0 \le \phi < \alpha)$$

$$b_o + \sum_1^{\infty} b_m \cos m\phi = 0 , \quad (\alpha < \phi \le \pi) .$$

The analysis proceeds as before. Instead of (19) we have

$$\frac{1}{2}\left\{\frac{1}{2}R_o(\mu) - \cos s\right\}\int_o^{\pi} f(\mu)\frac{du}{d\xi}\,d\xi + \left\{\frac{1}{2}R_1(\mu) - 1\right\}\int_o^{\pi} f(u)\frac{du}{d\xi}(t + s\cos\xi)d\xi(t + s\cos x)$$

$$+ \sum_1^{\infty}\frac{1}{m}\int_o^{\pi} f(u)\frac{du}{d\xi}\cos m\xi d\xi \cos mx = 0, \quad (0 < x < \pi).$$

Instead of (20) we have to assume

$$f(u)du/d\xi = C(1 + d\cos\xi) ,$$

where d is a constant to be determined later. Then

$$\frac{1}{2}\left\{\frac{1}{2}R_o(\mu) - \ell n\, s\right\} + \left\{\frac{1}{2}R_1(\mu) - 1\right\}(t + \frac{1}{2}sd)(t + s\cos x) + \frac{1}{2}d\cos x = 0 .$$

The term independent of x and the coefficient of cos x must both be zero. On eliminating d between the resulting equations we find the following equation for μ, which is a generalization of (22):

$$(24) \qquad R_o(\mu) + \frac{4t^2K}{1+s^2K} = 2\,\ell n\, s, \qquad K = \frac{1}{2}R_1(\mu) - 1 .$$

In a similar way a third approximation can be obtained by including the first two terms of the second infinite series in (11a). We quote the equation for μ in the form

$$(25a) \qquad R_o(\mu) + \frac{4t^2K}{1+s^2K}(1+\delta) = 2\,\ell n\, s,$$

where

$$(25b) \qquad \delta = \frac{L\left\{(2-3t)+s^2K(2+t)\right\}^2}{2K\left\{(1+s^2K)(1+s^4L)+8s^2t^2L\right\}} , \qquad L = R_2(\mu) - 1 .$$

Obvicusly we can generalize to any order of approximation but the algebra rapidly becomes more complicated for higher orders, if explicit equations like (24), (25) are required.

V. Results

We first of all examine the accuracy of the approximation (13b). Table I gives numerical values for $1/2\ m\ R_m(r)$ for $\mu = 0(0.2)1$. This verifies that (13b) is approximately true. The accuracy is better for μ near zero, as we should expect from (13a).

Table I

μ	0.00	0.20	0.40	0.60	0.80	1.00
$R_0(\mu)$	$-\infty$	- 7.542	- 2.751	- 1.223	- 0.471	0.000
$\frac{1}{2}R_1(\mu)$	1.000	1.105	1.298	1.704	2.947	∞
$R_2(\mu)$	1.000	1.029	1.070	1.129	1.212	1.333
$\frac{3}{2}R_3(\mu)$	1.000	1.013	1.031	1.055	1.086	1.125

Numerical results for the exponent μ as a function of the sector semi-angle α are given in Table II. From Table I it is clear that $R_1(\mu)$ is large when μ is near unity. It might therefore seem that the first approximation obtained in Section 3 by neglecting $R_1(\mu)$ would not be satisfactory for μ near unity. But (24) shows that $R_1(\mu)$ appears in both top and bottom of the correction term. When μ is near unity the correction term is approximately $4t^2 = 4\cos^4\frac{1}{2}\alpha$ which is very small when α is near π. Hence (22) is a good approximation for μ near both zero and unity. The first and second approximations are plotted in Fig. 1.

A third correction could be obtained from (25). The size of δ compared with unity indicates the magnitude of the correction term. For $\alpha = \frac{1}{2}\pi$ the first approximation gives 0.56_9, the second 0.50_5, and $\delta \approx 0.06$. Hence the third approximation would be 0.50_1, compared with the exact 0.500. The quantity δ is small for α between 0.2π and 0.4π. It tends to roughly 0.2 as α tends to zero and to 0.5 as α tends to π, but in both these regions the difference between first and second approximations is small. Approximately $\delta = 0.18, 0.35$ for $\alpha = 0.6\pi, 0.7\pi$ respectively.

The results for quarter-planes are as follows. For $\Psi = 0$ on a quarter-plane $\mu \approx 0.30$. For $\partial\Psi/\partial n = 0$ on a quarter-plane ($\Psi = 0$ on a plane

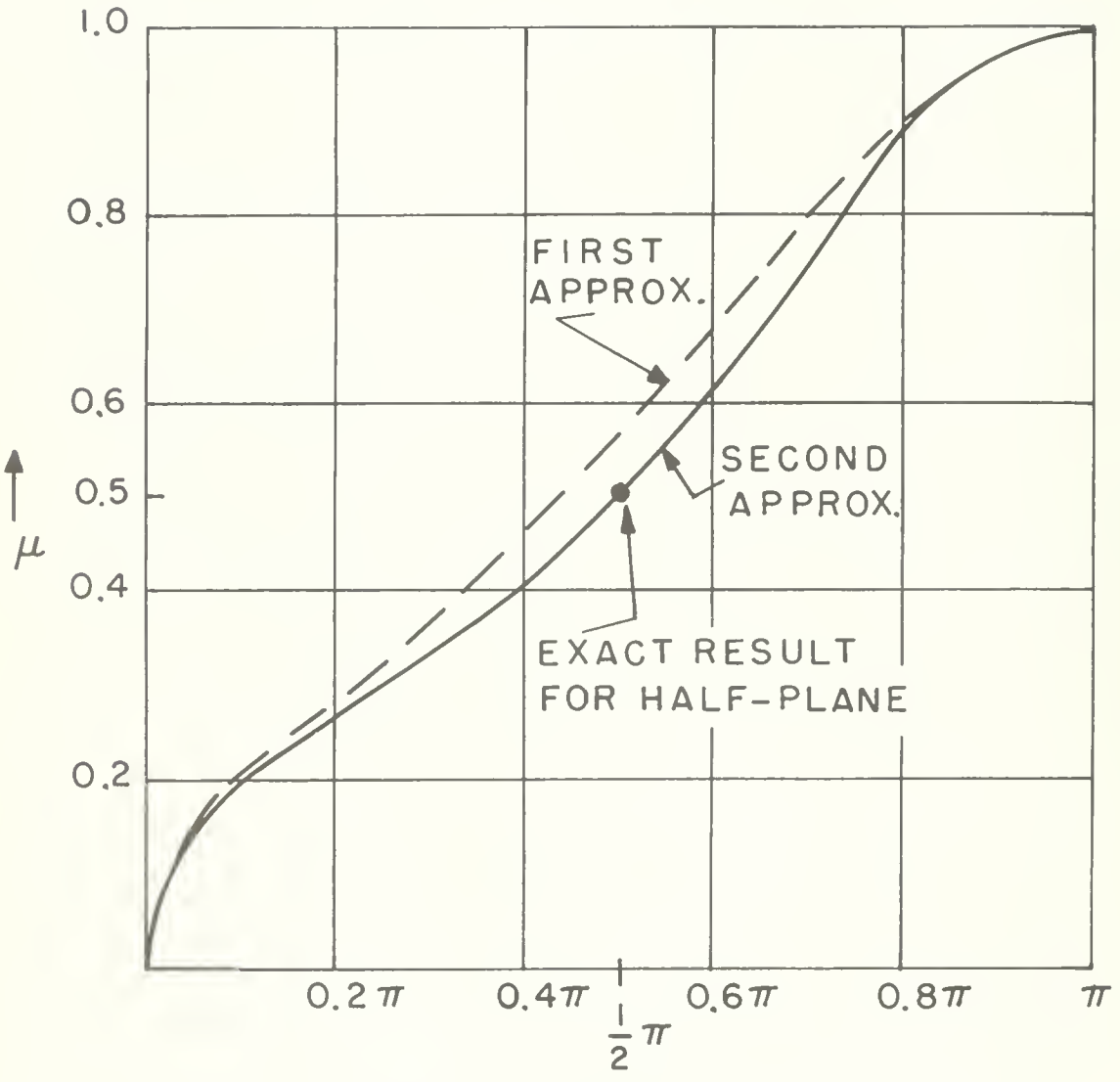

FIG. 1 GRAPH OF EXPONENT μ vs. SEMI-ANGLE α

$\left[\psi \sim r^{\mu} \text{ as } r \longrightarrow 0 \text{ for fixed } \theta, \phi\right]$

with $\alpha = 3/4\ \pi$) we find $\mu \approx 0.82$.

Numerical results are given in [5] but there seems to be some confusion of notation. In Table II we quote some results from [5] on the assumption that the conditions $\Psi = 0$ and $\partial\Psi/\partial n = 0$ have become interchanged in the numerical working, although this seems unlikely from an examination of the analysis. In any case Kraus' results seem to indicate a discontinuity in μ at $\alpha = \pi/2$ which is very unlikely.

Table II

α/π		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
First Approx	(22)	0.00	0.20_3	0.28_2	0.36_7	0.46_2	0.56_9	0.68_5	0.80_4	0.90_6	0.97_6	1.00
Second Approx	(24)	0.00	0.19_5	0.26_4	0.33_2	0.41_0	0.50_5	0.62_0	0.75_3	0.89_2	0.97_5	1.00
$\mu \approx 0$	(26)	0.00	0.19_7	0.27_0	0.34_6	0.43_2	0.53_5	0.66_4				
$\mu \approx 1$	(27)							0.60_5	0.77_9	0.90_1	0.97_5	1.00
[5]	(?)	0.00	0.22	0.33	0.42	0.48	0.50	0.65	0.80	0.90	0.95	1.00

Equation (22) can be solved approximately for μ near zero in the following way. From (12), for small μ,

$$R_o(\mu) \approx -(2/\mu)\left\{1 - \mu\ 2\ \ell n\ 2\right\} .$$

$R_o(\mu)$ is large for small μ, α is small, and (22) gives

$$\mu = -(2\ \ell n\ \tfrac{1}{4}\ \alpha)^{-1} . \qquad (26)$$

As previously noted, for μ nearly equal to unity it might seem that the first approximation in Section 3 which ignores $R_1(\mu)$ would not be satisfactory and that (24) should be used instead. If we set $\mu = 1 - \eta$, $\alpha = \pi(1-\varepsilon)$ then

$$R_0(\mu) \approx -2\eta\,, \qquad R_1(\mu) \approx \eta^{-1}$$
$$\sin\tfrac{1}{2}\alpha \approx 1 - \tfrac{1}{8}\pi^2\varepsilon^2\,, \qquad \ell n\, s \approx -\tfrac{1}{4}\pi^2\varepsilon^2$$
$$\cos\tfrac{1}{2}\alpha \approx \tfrac{1}{2}\pi\varepsilon\,, \qquad t \approx \tfrac{1}{4}\pi^2\varepsilon^2\,.$$

Equation (24) gives

$$-2\eta + \tfrac{1}{4}\pi^4\varepsilon^4 = -\tfrac{1}{2}\pi^2\varepsilon^2$$

For small η the second term is of lower order than the others and we find

(27) $$\mu \approx 1 - \tfrac{1}{4}(\pi-\alpha)^2$$

This formula would have been obtained directly from (22) and this confirms that (22) is a good approximation for μ nearly equal to unity as well as for μ near zero.

Dr. L.B. Felsen suggested to me that it would be of interest to compare the results for α near 0 and π with the corresponding results for small-angle cones. The following approximations can be derived from the expression for $P_\mu(\cos\theta)$ in [4] p.225 (53): If μ and θ are small,

(28a) $$P_\mu(-\cos\theta) \approx 2\mu \ln(\sin\tfrac{1}{2}\theta) + 1\,.$$

If $\mu = 1 - \delta$ and δ, θ are small,

(28b) $$P_\mu^1(-\cos\theta) = -\,dP_\mu(-\cos\theta)/d\theta \approx -\left\{2\delta\theta^{-1} + \theta\right\}\,.$$

Consider a cone with axis lying along $\theta = \pi$ in a spherical polar coordinate system (r, θ, ϕ). The cone is oriented in a different way from the sector considered earlier. If the cone lies in $\theta = \pi - \beta$ when β is small, and Ψ is axially symmetric, the eigenvalues for the region $0 \le \theta \le \pi - \beta$ are given by the values of μ for which

$$P_\mu(-\cos\beta) = 0\,,$$

i.e. from (28a), $$\mu \approx -(2 \ln\tfrac{1}{2}\beta)^{-1}.$$

On comparing with (26) we see that the sector and the cone have the same eigenvalues if $\beta = \frac{1}{2}\alpha$. [For the same length of circumference for a given r we should have $2\pi\beta = 4\alpha$ i.e. $\beta = (2/\pi)\alpha$].

Consider next the case $\partial\Psi/\partial n = 0$ on the cone. From the nature of the mixed boundary conditions in (5) we require, in terms of the coordinates used for the cone, a solution such that $\Psi = 0$ on (say) $\phi = 0$. Hence we are looking for an eigenfunction of the form

$$P_\mu^1(\cos\theta)\sin\phi \ .$$

The eigenvalue is given by

$$\left[dP_\mu^1(\cos\theta)/d\theta\right]_{\theta=\pi-\beta} = 0 \ .$$

From (28b)

$$\delta = \frac{1}{2}\beta^2 \quad \text{i.e.} \quad \mu = 1 - \frac{1}{2}(\pi-\gamma)^2,$$

where $\gamma = \pi-\beta$. On comparing this result with (27) it is seen that the sector and the cone are equivalent if

$$\pi - \gamma = 2^{-1/2}(\pi - \alpha) \ .$$

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